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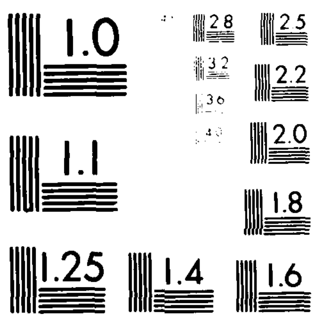

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ON TESTS FOR UNIFORMITY: NEYMAN'S STATISTIC AND STATISTICS BASED ON GAPS AND STRETCHES

By

Herbert Solomon and Michael A. Stephens

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On Tests for Uniformity: Neyman's Statistic and Statistics Based on Gaps and Stretches

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Introduction.

In this paper we provide percentage points for Neyman's goodness-of-fit statistics of order two for the uniform distribution. Recent work has suggested that this statistic is powerful against a wide range of alternatives. The statistic is a combination of the sample mean and sample variance of the observations. The percentage points have been found by fitting Pearson curves, following the work of F. N. David (1939) who gave the first four moments of the Neyman statistic. We then turn our attention to another situation concerned with sampling from a uniform distribution. Deken (1980) has produced exact distributions and moments for the largest gaps (spacings) and stretches (higher order spacings) among points uniformly distributed on a unit interval. An approximation to the distribution is also suggested by Deken. We develop the Pearson curve fit for the distribution of this maximum statistic and find it is excellent over the range of values and somewhat better than the approximation in the lower tail region. The test statistic developed by Deken is powerful against alternatives to the uniform distribution that are likely to produce several clusters among the n points along the line. Deken suggests multiple comparison testing as a motivation. Another possibility occurs for some physical phenomena, e.g. Poisson processes, where events are registered on a line and the issue at hand is usually whether there is uniformity which would represent one explanation, or clusters of the n points which would depict another model.

Neyman's Statistic N_2^2.

Neyman (1937) suggested that any density $f(x)$ on the interval $(0,1)$ can be written in the form

$$(1) \qquad f(x) = c \exp\{1 + \sum_{j=1}^{k} \theta_j \ell_j(x)\}, \quad 0 < x < 1, \quad k = 1,2,\ldots$$

where $\ell_1(x), \ell_2(x),\ldots$ are Legendre polynomials, $\theta_1,\theta_2,\ldots,\theta_k$ are parameters, and c, a function of $\theta_1,\theta_2,\ldots,\theta_k$, is a normalizing constant. When $\theta_j = 0$, for all $j \geq 1$, $f(x)$ is the uniform density $f(x) = 1$ written $U(0,1)$. The Legendre polynomials are orthogonal on the interval $(0,1)$, and, by varying k, $f(x)$ may be made to approximate any given alternative. As the θ_j increase, the density $f(x)$ varies smoothly from the uniform distribution; thus the test for uniformity can be put in the form of a test on the parameter values, i.e. a test of

$$H_0: \sum_{j=1}^{k} \theta_j^2 = 0 .$$

By likelihood ratio methods, Neyman found an appropriate statistic for testing H_0. Suppose $x_1,x_2,\ldots,x_n$ is the given random sample. For given k the test statistic is N_k^2, calculated as follows:

(a) Calculate

$$(2) \qquad v_j = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \ell_j(x_i) , \quad j = 1,\ldots,k ;$$

(b) Then

$$(3) \qquad N_k^2 = \sum_{j=1}^{k} v_j^2 .$$

In these calculations, $\ell_j(x)$ is best expressed in terms of $y = x-0.5$. For the first four polynomials

$$\ell_1(x) = 2\sqrt{3}y; \ \ell_2(x) = \sqrt{5}(6y^2-0.5);$$

$$\ell_3(x) = \sqrt{7}(20y^3-3y); \ \ \ell_4(x) = 3(70y^4-15y^2+0.375) \ .$$

In general, H_0 will be rejected for large values of N_α^2. Note that N_1 is equivalent to $\bar{x}$, the mean of the x_i. In fact $N_1^2 = v_1^2$ and $v_1 = (12n)^{1/2}(\bar{x} - 0.5)$. Then let t_α be the upper tail percentage point for N_1^2 at significnce level α, and let $Z_{\alpha U}$, $Z_{\alpha L}$ be the upper and lower tail percentage point at level α for $\bar{x}$; we have $Z_{\alpha U} = 1-Z_{\alpha L}$, and $t_{2\alpha} = 12n(Z_{\alpha U} - 0.5)^2 = 12n(0.5 - Z_{\alpha L})^2$. Thus significance points for N_1^2 can be found from significance points for $\bar{x}$; a table of such points is available for example, in Stephens (1966, Table 1). Further, $v_2 = \Sigma(x_i - 0.5)^2/n = s^2$, a form of sample variance, and so N_2^2 is a combination of both $\bar{x}$ and s^2. In this paper we concentrate on N_2^2.

Neyman showed that, on H_0, the v_j are asymptotically independent, and each is normally distributed with mean 0 and variance 1. Thus the asymptotic null distribution of N_k^2 is χ_k^2; for the alternative family (1) the asymptotic distribution is noncentral χ_k^2. David (1939) examined the null distribution of $N_1^2 = v_1^2$ and $N_2^2 = v_1^2 + v_2^2$ for finite n, by calculating their moments and fitting Pearson curves, David showed that for $n \geq 20$, the χ_k^2 approximations were very good for N_k^2, when $k = 1$ or 2. The tests are consistent and asymptotically unbiased.

Since Neyman's early work many tests for uniformity have been developed, and the statistic has been somewhat overlooked. In that era before computers the statistics also required much computation, as was pointed out by David (1939). Recently, however, Locke and Spurrier (1978) have made extensive Monte Carlo studies of various tests for uniformity and have shown that N_2^2 is effective against a wide range of alternatives. We have also shown this to be so, in an unpuboished Monte Carlo study with some alternatives the same as, and some different from those of Locke and Spurrier. The fact that N_2^2 uses both sample mean and sample variance makes it plausible that it will detect many types of non-uniformity, and these simple statistics also have a natural appeal. It seems worthwhile therefore to give a set of upper tail percentage points for N_2^2, on H_0, for small values of n. This is done in Table 1. The points are derived by fitting Pearson curves to the first four moments given by David (1939). These moments are as follows:

$$\mu = 2$$

$$\mu_2 = 4 - \frac{32}{35n}$$

$$\mu_3 = 16 + \frac{704}{49n} - \frac{722208}{35035n^2}$$

$$\mu_4 = 144 + \frac{15216}{49n} - \frac{2203468}{35035n^2} + \frac{17946980}{119119n^3} .$$

David used two moments, or three moments with a zero start to approximate the distribution of N_2^2 by Pearson curves of Type I for N_1^2 and Type VI for N_2^2 for n = 5, 10, 20, 30, 50, 100, and concluded that these approximations gave very good results. A table of percentage points was not given, however, (it would have been very tedious to calculate at the time) and the present Table 1 might therefore be regarded as an extension of David's work, making use of modern capabilities. The Pearson curve approximations are based on the extensive tables of significance points produced by Johnson, Nixon, Amos an Pearson (1963) and reproduced in Pearson and Hartley (1972). David showed that the asymptotic χ_2^2 approximation will be accurate for quite small n, and Table 1 demonstrates this also. The last row of percentage points is obtained from χ_2^2. Further comments on the Neyman tests are in Pearson (1938) and David (1939).

Barton (1953) considered a slightly different class of alternatives given by

$$f(x) = \sum_{j=0}^{k} \theta_j \ell_j(x) , \quad 0 \leq x \leq 1 , \quad k = 0,1,\ldots$$

with θ_0 equal to 1. A restriction must now be placed on the θ_i to ensure that the density is always positive. The same statistics N_k^2 may again be used to test for uniformity against this alternative.

TABLE 1

Upper tail percentage points for N_2^2 .

$N \diagdown \alpha$	0.5	0.75	0.8	0.9	0.95	0.975	0.99	0.995
2	1.571	2.702	3.058	4.193	5.411	6.748	8.741	10.454
3	1.492	2.685	3.069	4.292	5.594	6.998	9.041	10.749
4	1.459	2.689	3.086	4.352	5.688	7.109	9.143	10.810
5	1.442	2.696	3.102	4.393	5.745	7.172	9.190	10.827
6	1.432	2.704	3.116	4.422	5.784	7.212	9.215	10.826
7	1.425	2.710	3.126	4.445	5.812	7.239	9.227	10.815
8	1.420	2.716	3.135	4.462	5.833	7.257	9.231	10.798
9	1.416	2.721	3.143	4.476	5.849	7.272	9.235	10.787
10	1.413	2.725	3.149	4.487	5.862	7.283	9.235	10.773
12	1.409	2.731	3.159	4.505	5.883	7.300	9.237	10.755
14	1.406	2.736	3.166	4.517	5.897	7.311	9.235	10.735
16	1.403	2.740	3.172	4.527	5.908	7.319	9.233	10.720
18	1.402	2.744	3.177	4.536	5.918	7.327	9.235	10.716
20	1.400	2.746	3.181	4.542	5.925	7.332	9.234	10.706
25	1.398	2.751	3.188	4.554	5.937	7.341	9.230	10.684
30	1.396	2.755	3.193	4.562	5.947	7.348	9.230	10.677
35	1.395	2.757	3.196	4.568	5.952	7.352	9.226	10.662
40	1.394	2.759	3.199	4.573	5.958	7.357	9.230	10.666
45	1.393	2.760	3.201	4.576	5.961	7.357	9.221	10.645
50	1.392	2.762	3.203	4.579	5.964	7.360	9.223	10.646
60	1.391	2.763	3.206	4.584	5.969	7.364	9.224	10.644
80	1.390	2.766	3.209	4.589	5.974	7.367	9.218	10.627
100	1.390	2.768	3.212	4.592	5.979	7.370	9.220	10.626
∞	1.386	2.773	3.219	4.605	5.991	7.378	9.210	10.597

Gaps and Stretches.

In an interesting paper, Deken (1980) has looked into the distribution of gaps and stretches that arise in sampling from a uniform distribution over the unit interval. For densities other than the uniform an appropriate probability inverse transformation can be employed to achieve a uniform distribution. Consider the order statistics $y_1, y_2, \ldots, y_n$, in a sample of size n from the uniform distribution. Define the p-stretches $z_1, z_2, \ldots, z_{n+1-p}$ as $z_j = y_{j+p-1} - y_j$, $j = 1, \ldots, n+1-p$. The variables z_j are called spacings ($p = 2$), or higher order spacings ($p > 2$). There is an extensive literature on spacings and in some cases this literature deals with the classical geometrical probability problems of random coverage of the circumference of a circle by random arcs. There is a duality between distributions related to spacings and distributions of coverage. A recent article by Holst (1980), gives some new results in this interesting subject as well as many references.

In some recent work on multiple comparisons, Welsh (1977), looks into the variables z_j and labels them gaps for $p = 2$ and stretches for $p > 2$. Deken's key contribution is to deal directly with the lack of independence between successive p-stretches. This is rather formidable for $p > 2$ and had led in the past to asymptotic considerations. The multiple comparison situation is one in which asymptotic results may not be sufficiently accurate and in which the number of points n may often be small. Deken demonstrates through a recursive formulation how to derive exact results in many cases. Specifically, he computed the exact distribution for the maximum p-stretch for all values of p for ten or

fewer points uniformly distributed in the unit interval. In addition to the exact distributions, he provides formulas for moments for $n = 2,3,\ldots,10$, $p = 2,3,\ldots,10$ and quantiles of the distribution. He also produces an approximation based on an independence assumption for the successive p-stretches utilizing the fact that the distribution of any individual p-stretch is Beta when sampling is from the uniform distribution. Therefore, an approximation to the exact distributions computed by Deken is that of the maximum of $(n+1-p)$ independent Beta variable.

In Table 2 quantiles are listed for several cases where each cell lists the exact value, the Pearson curve fit, and the approximation given by the Beta assumption. For the column where the number of points is five and the stretch is five and similarly, where the number of points is ten and the stretch is ten, no approximate values are given because we are dealing directly with the distribution of the range. The Pearson curve fits do extremely well over all cells, whereas the Beta approximation is only viable for the upper tail.

From Deken's development, it appears that for $n > 10$, the analytical development of moments is more feasible than producing the quantiles of the exact distribution of the maximum p-stretches. Thus, should moments become available for $n > 10$, the PC fit can be achieved rather econom. .lly and with the knowledge that the fit will be excellent.

TABLE 2

Percentage points for the maximum p-stretch in samples of size n from the uniform distribution.

	α	n=5 p=2	n=5 p=3	n=5 p=4	n=5 p=5	n=10 p=9	n=10 p=10
Exact		.1570	.2308	.2897	.3426	.5635	.6058
P.C.	.05	.1850	.2321	.2903	.3426	.5637	.6059
Approx.		.1202	.2506	.3425		.6316	
Exact		.1917	.2811	.3525	.4161	.6176	.6632
P.C.	.10	.1904	.2811	.3527	.4161	.6176	.6632
Approx.		.1523	.2963	.3993		.6738	
Exact		.2555	.3716	.4649	.5458	.7033	.7526
P.C.	.25	.2530	.3700	.4645	.5458	.7032	.7527
Approx.		.2178	.3813	.5000		.7414	
Exact		.3340	.4754	.5914	.6862	.7878	.8377
P.C.	.50	.3347	.4762	.5915	.6862	.7878	.8377
Approx.		.3076	.4854	.6144		.8098	
Exact		.4254	.5839	.7100	.8062	.8577	.9036
P.C.	.75	.4280	.5852	.7102	.8063	.8578	.9035
Approx.		.4135	.5944	.7230		.8686	
Exact		.5218	.6832	.8024	.8878	.9069	.9455
P.C.	.90	.5207	.6814	.8022	.8878	.9069	.9455
Approx.		.5181	.6905	.8089		.9118	
Exact		.5837	.7393	.8488	.9236	.9301	.9632
P.C.	.95	.5795	.7364	.8484	.9235	.9300	.9633
Approx.		.5821	.7445	.8527		.9329	
Exact		.6983	.8311	.9159	.9673	.9621	.9845
P.C.	.99	.6947	.8316	.9159	.9672	.9616	.9847
Approx.		.6982	.8333	.9171		.9630	

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Distributions of two statistics arising in sampling from a uniform distribution are investigated. They are Neyman's smooth goodness-of-fit statistic of order two and the maximum of a gap or stretch between p points in a sample of size n, $p \leq n$. < or =
Pearson curve fits are applied to each situation thereby extending percentile tables provided by F.N. David (1939) for Neyman's statistic of order two, and indicating usage in computing percentiles for the stretch statistic. The Pearson curve values are excellent approximations along the range of values of each statistic.

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